Pole solutions in the case of problems of flame front propagation and Saffman-Teylor "finger" formation without surface tension: open problems and possible ways of their solutions.

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Some physical problems as flame front propagation or Laplacian growth without surface tension have nice analytical solutions which replace its complex integro-differential motion equations by simple differential equations of poles motion in a complex plane. Investigation of these equation was the main topic of Kupervasser Oleg Ph.D. Thesis[7]. Some very interesting open problems were hanged up there.Here we give these open problems and possible ways of their solutions.

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In the beginning let us to consider the case of Saffman-Teylor "finger" formation.

1) The case of Laplacian growth in the channel without surface tension was in details considered by Mark Mineev-Weinstein and Dawson [1]. In this case the problem has the beautiful analytical solution. Moreover they assumed that all major effects in the case with vanishingly small surface tension may be received also without surface tension. It would allow applying to vanishingly small surface tension case the powerful analytical methods developed for the no surface tension case. However without additional assumptions this hypothesis may not be accepted.

The first objection is related to finite time singularities for some initial conditions. Actually, for overcoming this difficulty the regular item with surface tension was introduced. This surface tension item is resulting in loss of the analytical decision. However regularization may be carried out much more simply - simply by rejecting the initial conditions which result to these singularities.

The second objection is given in work Siegel and Tanveer [2]. There it is shown, that in numerical simulations in a case with any (even vanishingly small) surface tension any initial thickness "finger" extends up to $\frac{1}{2}$ thickness width of the channel. The analytical solution in a case without a surface tension results in constant thickness of the "finger" equal to its initial size that may be arbitrary. Siegel and Tanveer however did not take into account the simple fact, that numerical noise introduces small perturbation or to the initial condition, or even during "finger" growth, which is equivalent to the remote poles, and with respect to this perturbation the analytical solution with constant "finger" thickness is unstable. By Mark Mineev-Weinstein [3] it was shown, that similar pole perturbations can give, at the some initial conditions, extending up to the Siegel and Tanveer solutions. This positive aspect of the paper [3] was mentioned by Sarkissian and Levine in them comment [4]. Summing up, it is possible to tell, that for identity of the results with and without surface tension it is necessary to introduce a permanent source of the new remote poles: it may be either external noise or infinite number of poles in an initial condition. What from these methods is preferred it is a open question yet. In the case of flame front propagation it was shown [7], that external noise is necessary for an explanation of flame front velocity increase with the sizes of system: the infinite number of poles in an initial condition can not give this result. It is interesting to know, what is situation in the channel Laplacian growth.

One of main results of Laplacian growth in the channel with a small surface tension is Saffman-Teylor "finger" formation with the thickness equal to $\frac{1}{2}$ thickness of the channel. And to use the analytical result received for zero surface tension, it is necessary to prove, that formation of the "finger" with thickness equal to $\frac{1}{2}$ thickness of the channel takes place without surface tension also. In our teamwork with Mark Mineev-Weinstein [5] it was shown, that for finite number of poles at almost all allowed (in the sense of not approaching to finite time singularities) initial conditions, except for small number of some degenerated initial conditions, they have asymptotic as some "finger" with any possible thickness. It should be mentioned, that the solutions and asymptotic found in [5] for finite number of poles are though also idealization, but quite have real sense for any finite intervals of time between appearance of the new poles introduced into system by external noise or connected to an entrance to the system of remote poles of an initial condition, including infinite number of such poles. The theorem proved in [5] and may be again applied for this final set of new and old poles is again received asymptotic, being again "finger", but already with possible new, distinct from former, thickness. Thus, introduction of a source of new poles results only in possible drift of thickness of the final "finger", but not changing of type of this solution. It should be mentioned, that instead of periodical boundary conditions, much more realistic physical boundary condi-

tions may be introduced [6], forbidding a stream through a wall which insert additional, probably useful, restrictions on a positions, number and parameters of new and old poles (explaining, for example, why the sum of all complex parameters α_i for poles give the real value α for the pole solution (5) in [3]), not influencing, however, as shown in [7], on correctness and applicability proved in [5] results and methods of their including. Given in [3] by Mark Mineev-Weinstein "proof", that steady asymptotic for Laplacian growth in a channel with zero surface tension is single "finger" with thickness equal to $\frac{1}{2}$ thickness of the channel, is unequivocally erroneous: completely the same method which was used in [3] to prove and demonstrate instability of "finger" with thickness distinct from $\frac{1}{2}$ with respect to introducing the new remote poles, instability of "finger" with thickness equal to $\frac{1}{2}$ may be proved and demonstrated! This objection was repeatedly stated to Mark Mark Mineev-Weinstein before the publication of his paper [3], however has not found any answer there. Moreover, in our teamwork [5] was is shown, that for finite number of poles any thickness "finger" is possible as asymptotic. It does not mean, nevertheless, that privileged role of "finger" with thickness $\frac{1}{2}$ cannot be proved in the case of surface tension absence, but means only that such the proof are not given in [3]. Let us try to give these correct arguments here. The general pole solution (5) in work [3] is characterized by the real parameter α being the sum of the complex parameters α_i for poles. Thickness of the asymptotic finger is simple function of α: ($Thickness = 1 - \frac{\alpha}{2}$). The value ($\alpha = 1$) corresponds to thickness $\frac{1}{2}$. As far as possible thickness of the "finger" is between 0 and 1, possible α value is in an interval between 0 and 2: ($0 < \alpha < 2$). The value $\alpha = 1$ corresponding to thickness $\frac{1}{2}$ is exactly in the middle of this interval. What occurs to quite possible initial pole conditions with α outside of limits from 0 up to 2? They are "not allowed" because of already known to us finite time singularities [5]. Also a part of solutions inside of interval $0 < \alpha < 2$ results to the similar finite time singularities. Exact necessary conditions, whether defining the initial pole condition as "allowed", i.e. singular, is still a open problem. How number of these "allowed" initial pole conditions (to be exact speaking, their percent from the full number of the possible initial pole conditions corresponding to the given real value α) is distributed inside of this interval? From the reasons of a continuity and symmetry with respect to $\alpha = 1$ it is possible to conclude, that this distribution has a minimum in point $\alpha = 1$ (thickness $\frac{1}{2}$!), the value which is the most remote from both borders of interval $0 < \alpha < 2$, being increased to borders $\alpha = 2$ or 0, and reaching 100 from all pole solutions outside of these borders. I.e. thickness $\frac{1}{2}$ is the most probable because for this thickness value the minimal percent of potentially capable to give such thickness value initial conditions is "not allowed", i.e. results to singularities. Source of new poles results to the drift of finger thickness, but this thickness drift is closed to the most probable and average size equal to $\frac{1}{2}$! The similar result is obtained in the case of Saffman-Teylor "finger" with vanishingly small surface tension and with some external noise. As it was desirable to be proved. Similar idea, that the initial conditions resulting to singularities, can provide the selected and special role of thickness $\frac{1}{2}$ solution stated by Procaccia [8] also. These given arguments are not, certainly, the strict proof, but only specifies a way to it. The inquisitive reader is invited to make and publish it.

Let us pass to a problem of flame front propagation.

2) For a cylindrical case of the flame front propagation problem at absence of noise (only numerical noise) (look [7] and the bibliography there) by Sivashinsky with help of numerical methods it was shown, that the flame front is continuously accelerated. During all this account time it is not visible any attributes of saturation. To increase time of the account is a difficult task. Hence, absence or presence of velocity saturation in a cylindrical case, as consequence of the flame front motion equation it is still a open problem yet.

For the best understanding of dependence of flame front velocity as functions of its radius in a cylindrical case similar dependence of flame front velocity on width of the channel (in a flat case) also was analyzed by numerical methods. Growth of velocity is also observed and at absence of noise (only numerical noise!) also any saturation of the velocity it is not observed. Introduction obvious Gaussian noise results to appearance of a point of saturation and its removal from the origin of coordinates with decreasing of noise amplitude, allowing extrapolating results on small numerical noise. (Fig. 2.6 in [7])

Hence, introducing of Gaussian noise in numerical calculation also for a cylindrical case can again results to appearance of a saturation point and will allow to investigate its behavior as function of noise amplitude by extrapolating results on small numerical noise. The Inquisitive Reader loved by me is invited again to make it and to publish the received interesting results.

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